

# Influence of the back reaction of the Hawking radiation upon black hole quasinormal modes

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## Abstract

We consider the BTZ black hole surrounded by the conformal scalar field. Within general relativity, the resonant *quasinormal* (QN) modes dominate in the response of a black hole to external perturbations. At the same time, the metric of an evaporating black hole is affected by the Hawking radiation. We estimate the shift in the quasinormal spectrum of the BTZ black hole stipulated by the back reaction of the Hawking radiation. For the case of the 2+1 dimensional black hole the corrected (by  $\sim \hbar$ ) metric is an *exact* solution [C.Martines, J.Zanelli (1997)]. In addition, in this case quantum corrections come only from matter fields and no from graviton loops, that is, one can solve the problem of influence of the back reaction upon the QN ringing self-consistently. The dominant contribution to the corrections to the QNMs is simply a shift of  $\omega^2$  proportional to  $-(\frac{\Lambda}{M})^{3/2}(4L^2 + M)\hbar$ . It is negligible for large black holes but essential for small ones, giving rise to considerable increasing of the quality factor. Thus, the small evaporating black hole is expected to be much better oscillator than a large one.

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In classical regime a black hole does not emit anything. It is characterized by its three parameters: mass, charge, and angular momentum. When perturbing a black hole the background geometry undergoes damping oscillations dominated at late times by the so called *quasinormal* modes. They are of a great importance because they depend upon the above parameters of a black hole only and not on the way of excitation. Thus these modes represent the characteristic resonance spectrum of a black hole response (see [1] for a review). The QN modes have gained considerable interest owing to their interpretation in ADS/CFT correspondence [2], [3], [4], [5], [6], and Loop Quantum Gravity (see for instance [7] and references therein). Recent investigation of black holes within brane models stimulated the calculation of QN modes of different higher dimensional black holes (see [8] and references therein). In addition, the QN radiation of dilaton black holes have been recently studied in [9].

At the same time a black hole radiates energy with thermal spectrum, when taking into account the effect of quantized fields near the black hole. Thus a black hole can exist in a thermal equilibrium with a heat bath composed of quantum fields interacting with the black hole geometry [10]. In four dimensions the back reaction problem is solved usually as follows: one considers the expectation value of the renormalized (approximate) stress-energy tensor in appropriate "vacuum" state [11] as the source in the Einstein equations and solves these equations self-consistently for the metric [12].

In four dimensions the corrected metric diverges at large  $r$ , and in order to restrict cumulative effect from the corrected geometry, one need to put a shell outside of which the geometry is "uncorrected". As a result the final metric inside the shell contains a constant, which is determined by boundary conditions at the shell. The latter is assumed to be posed at some fixed distance from the event horizon. The picture significantly depends upon this boundary condition at the shell. Thus the two models are generally accepted. First, when one specifies the total energy of the system at the shell. That is micro-canonical ensemble. The second choice, a canonical ensemble, is to fix the temperature at the shell (see [13] for re-

cent references). If one would like to find the QN modes of such "corrected" metric one have to deal with a step-function (or delta-function) in the corresponding effective potential, at the radius of the shell  $r_0$ . This delta- function would crucially change the eigenvalues to be determined. The search of QN modes for such a "dirty" black hole should be done in a model independent way, in order, for example, that the found modes would not depend upon  $r_0$ . What is even more important in four dimensions, that if taking into consideration corrections from quantized fields of order  $\hbar$ , one must include corrections of the same order coming from quantum gravity.

Fortunately, in 2+1 dimensions the situation is much easier. First of all, the 2+1 gravity has no propagating degrees of freedom and at each point the Riemann tensor is completely determined by the matter source there. A quantum gravity in 2+1 dimensions is renormalizable and finite [14]. Thus the only radiative corrections to the geometry are coming from quantum excitation of the matter fields, and, the perturbative expansion receives no corrections from graviton loops [14]. At the same time there is a useful black hole solution in three dimensions with negative cosmological constant, the Banados-Teitelboim-Zanelli (BTZ) black hole [15].

The QN behavior of asymptotically anti-de Sitter (ADS) black holes [2], [5] crucially depends upon the black hole size relative to the ADS radius: for large BHs the QN modes are proportional to the radius of a black hole [2], while for small black holes they approach the modes of the empty ADS space-time [3]. The ADS space-time forms an effective confining box, and the potential diverges at spacial infinity. The case of conformal scalar field is different since the potential approaches a constant at infinity. That is why the QN behavior of conformal scalar field is different from that of the "ordinary" minimally coupled field studied in [2], [3], [4]. The QN modes of the BTZ black hole were calculated for conformal scalar field in [16], and for non-conformal scalar, electromagnetic and dirac fields in [6].

Consider the system consisting of the BTZ black hole and the conformal scalar field surrounding it. Let us find out what will happen with QN modes which govern the decay of this conformal scalar field

if taking into account the back reaction of quantum radiation of the same field upon the surrounding geometry. For this case and transparent boundary conditions at infinity the stress-energy tensor  $\langle T_{\mu\nu} \rangle$  was calculated in [17]. The  $O(\hbar)$  correction to the black hole geometry due to the radiative conformal field is governed by the semiclassical equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa \langle T_{\mu\nu} \rangle. \quad (1)$$

An *exact* solution of these equations was found by Martines and Zanelli in [18]:

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2d\theta^2, \\ f(r) = \left( r^2\Lambda - M - \frac{2l_p F(M)}{r} \right), \quad (2)$$

Here  $F(M)$  is determined in the following way [17]:

$$F(M) = \frac{M^{3/2}}{2\sqrt{2}} \sum_{n=1}^{\infty} e^{-in\delta} \frac{\text{Cosh}[2\pi n\sqrt{M}] + 3}{(\text{Cosh}[2\pi n\sqrt{M}] - 1)^{3/2}}, \quad (3)$$

where  $\delta$  is an arbitrary phase.

We used  $G = 1/8$ ,  $M$  can be associated with the black hole mass [19],  $l_p = \hbar/8$  is the Plank mass in three dimensions, the Plank mass  $m_p = \hbar/l_p = 8$  is independent of  $\hbar$ . The series (2) converges exponentially for any  $M > 0$ . For  $M \gg 1$  the first term dominates the series  $F(M) \sim e^{-\sqrt{-\pi M}} \rightarrow 0$  and the BTZ black hole is recovered. The metric (2) is an exact solution of the back reaction problem for the one-loop effective energy momentum tensor of a scalar field conformally coupled to gravity. Formally the metric (2) coincides with an exact solution for the BTZ black hole "dressed" by conformal scalar field [20].

Next, we shall consider the corrected metric (2) as a background for conformal scalar field and find the corresponding QN spectrum. The conformally coupled scalar wave equation has the form:

$$\nabla^2\Phi(t, r, \theta) = \frac{1}{8}R\Phi(t, r, \theta). \quad (4)$$

After the change of the wave function  $\Phi = \Psi/\sqrt{r}$ , and the radial coordinate  $dr^* = dr/f(r)$ , and, separation

of angular and time coordinates  $t$  ( $\Psi \sim e^{i\omega t}$ ) and  $\theta$  ( $\Psi \sim e^{iL\theta}$ ) one comes to the wave equation:

$$\left( \frac{d^2}{dr^{*2}} + \omega^2 - V \right) \Psi(r^*) = 0, \quad (5)$$

where the potential  $V$  has the following form

$$V = \left( \frac{M + 4L^2}{4r^2} - \frac{3l_p F(M)}{2r^3} \right) f(r). \quad (6)$$

In the considered range of parameter  $M$ , this potential as a function of  $r^*$  approaches its maximum at  $r^* = 0$  (spacial infinity) and goes to zero at  $r^* = -\infty$  (horizon) without any barriers near the black hole horizon (as it takes place for conformal scalar field around SAdS black hole [16]). Thus the effective potential of the quantum corrected BTZ black hole has the same features as that of the "pure" BTZ black hole.

In asymptotically flat space-time the QN modes are determined as the eigenvalues  $\omega$  such that, under the choice of the positive sign of the real part of  $\omega$ , QNMs satisfy the following boundary conditions

$$\Psi(r^*) \sim C_{\pm} \exp(\pm i\omega r^*), \quad r^* \rightarrow \pm\infty, \quad (7)$$

corresponding to purely in-going waves at the event horizon and purely out-going waves at spacial infinity. In our case the space-time is asymptotically anti-de Sitter and the appropriate boundary condition at spacial infinity is the Dirichlet one [21], while at the horizon it is, certainly, the requirement of purely in-going waves.

From here and on, in order to find the dominant contribution to the QN spectrum from  $O(\hbar)$  correction to the BTZ space-time, we shall neglect the order of  $l_p$  higher than first. Thus inverting the  $r$  coordinate as a function of  $r^*$  we find up to the first order of  $l_p$ :

$$r(r^*) = \sqrt{\frac{M}{\Lambda}} \frac{1 + \alpha^2}{1 - \alpha^2} + \frac{F(M)}{2M} \left( 2 - \frac{16(\ln \alpha + \ln(\alpha + 1))}{\alpha^2 + \alpha^{-2} + 2} \right) l_p + O(l_p), \quad (8)$$

where  $\alpha = e^{\sqrt{\Lambda M}r^*}$ . The  $r^*$  goes from  $-\infty$  to 0 as  $r$  goes from the event horizon to infinity. Thus the

value  $e^{\sqrt{\Lambda M}r^*}$  is always less than 1 and we can, following the paper [16], expand the effective potential into series of powers of  $\alpha$ . The first term, as it was shown in [16], gives the dominant QN behavior with good accuracy. Wishing to estimate *dominant* contribution to the shift of the QN spectrum we shall be restricted here by considering corrections greater than  $O(l_p^2, \alpha^4, \alpha^2 l_p)$ . For large black holes it is understood that next terms in  $\alpha^2$  are more important than even first correction  $\sim \hbar$ . Yet, in the regime of small black hole, where the back reaction is significant, the higher order corrections in  $\alpha^2$  is less important than  $\sim l_p$ -corrections. Thus, the approximated potential we shall investigate, has the form:

$$V(r^*) = \frac{(4L^2 + M)F(M)}{(\frac{M}{\Lambda})^{3/2}}l_p + \\ + (4L^2 + M)\Lambda e^{2\sqrt{\Lambda M}r^*} + O(l_p^2, \alpha^4, \alpha^2 l_p). \quad (9)$$

The QN modes for "uncorrected" potential  $V_0(r^*) = V_0 e^{2\sqrt{\Lambda M}r^*}$ ,  $V_0 = (4L^2 + M)|\Lambda|$ , were calculated in [16]. Comparison of the results obtained through  $V_0(r^*)$  with higher order corrections in  $\alpha$  shows that the dominant behavior is stipulated by this approximated potential  $V_0(r^*)$  [16]. In fact, the above potential  $V$  given by the formula (9) differs from  $V_0(r^*)$  only by a constant shift  $\frac{(4L^2 + M)F(M)}{(\frac{M}{\Lambda})^{3/2}}l_p$ , which simply can be thought of as a shift of  $\omega^2$ . Therefore, an exact solution of the wave equation (5) will have the similar form as that obtained in [16]. Namely, the Green function  $G(r^*, \xi; \tilde{\omega})$ , satisfying the wave equation

$$\left( \frac{d^2}{dr^{*2}} + \tilde{\omega}^2 - V \right) \tilde{G}(r^*, \xi; \tilde{\omega}) = -\delta(r^* - \xi), \quad (10)$$

has the form:

$$G(r^*, \xi < r^*; \tilde{\omega}) = \\ \frac{I_\nu(z(\xi)) [I_\nu(Z_0)K_\nu(z(r^*)) - K_\nu(Z_0)I_\nu(z(r^*))]}{\sqrt{\Lambda M}I_\nu(Z_0)}. \quad (11)$$

Here

$$\tilde{\omega}^2 = \omega^2 - \frac{(4L^2 + M)F(M)}{(\frac{M}{\Lambda})^{3/2}}l_p, \quad (12)$$

$$\nu = -i\tilde{\omega}/\sqrt{\Lambda M}, \quad z = \sqrt{\frac{V_0}{\Lambda M}}\alpha, \quad Z_0 = \sqrt{\frac{V_0}{\Lambda M}}.$$

The QN modes are the poles of this Green function and thereby are zeros of the modified Bessel function

$$I_\nu(Z_0) = 0 \quad (13)$$

We see (Fig.1, 2, 3) that this shift, being negligible for large black holes, becomes significant for small black holes and gives rise to increasing of the real oscillation frequency and to decreasing of the damping rate in this regime. Therefore the quality factor, which is proportional to  $|\omega_{Re}| / |\omega_{Im}|$ , is increasing considerably when one goes over to considering of smaller mass of the black hole and, at the same time, including the back reaction of the Hawking radiation. From this, one can conclude that the small evaporating black hole is expected to be much better oscillator than a large one. Remember, that the quality factor of the large Schwarzschild black hole is of order  $L$  at the fundamental overtone which is, for instance, roughly  $10^6$  times is less than that of an atom. That is, the large black hole is a very poor oscillator [22]. Note also, that for very small mass, next corrections in  $\hbar$  should be considered in the semiclassical equations.

The QN frequencies shown in figures 1, 2, 3 are found under the Dirichlet boundary conditions as closest to the  $\omega_{Re}$ -axis poles of the modified Bessel function. Nevertheless, the shift given by the formula (12) does not depend upon the boundary conditions to be chosen. The dependence on  $L$  of the QNMs is demonstrated on Fig.4. We see that both  $\omega_{Re}$  and  $\omega_{Im}$  are roughly proportional to  $L$ .

The influence of the back reaction on higher overtones is simply the above shift given by (12) and certainly is negligible for modes with huge imaginary part. The higher overtones can be found by extensive numerical search of the zeros of the modified Bessel function. The higher overtone behavior strongly depends upon the value of  $z$ : while first several overtones have both non-vanishing real and imaginary parts, the higher ones have tiny real parts, and the more  $z$ , the greater the number of modes with non-vanishing real part. Asymptotically, for highly damping modes, governed by an approximated potential (9), one has

$$Re\tilde{\omega} \rightarrow 0, \quad Im\tilde{\omega} \rightarrow n + z - 1 \quad \text{as} \quad n \rightarrow \infty. \quad (14)$$

Note, that this asymptotic regime comes very rapidly, i.e. it takes place already at fifth overtone for  $z = 3$  and at somewhat greater overtone number for greater  $z$ . This let us hope that the same asymptotic behavior will take place when considering complete effective potential with no approximations. This quick falling into the asymptotic regime repeats the high overtone behavior of non-conformal scalar field around ADS black hole [4].

Note that under the metric perturbations of the above mentioned conformally dressed black hole [20] there appear the physically accepted growing gravitational modes if imposing Dirichlet boundary conditions [23]. Even though this indicates upon classical instability of the black hole, the considered here spectral problem for the system, consisting of the black hole and the conformal scalar field, remains consistent since we are interested in study of decay of the scalar field only and there is no coupling with gravitational perturbations. For realistic 4-dimensional models such instability would certainly "cut off" the motivation of study of the QN spectrum. In three dimensions it is much more important that we have consistent quantum corrected solution allowing to avoid considering the problem in the realm of quantum gravity. After all, the obtained shift of  $\omega^2$  does not depend upon boundary conditions which are very controversial in anti de Sitter space-time [21, 16, 5, 24].

**Conclusion.** We have estimated the dominant contribution to the back reaction shift of the quasi-normal modes for BTZ black hole surrounded by conformal scalar field. It is interesting that the considering of the effect of back reaction on the metric gives rise to the sharp increasing of the quality factor of small black holes. This means that a small black hole is a much better oscillator than a large one, and, therefore, investigation of the resonance quasinormal spectrum for such black holes should be important.

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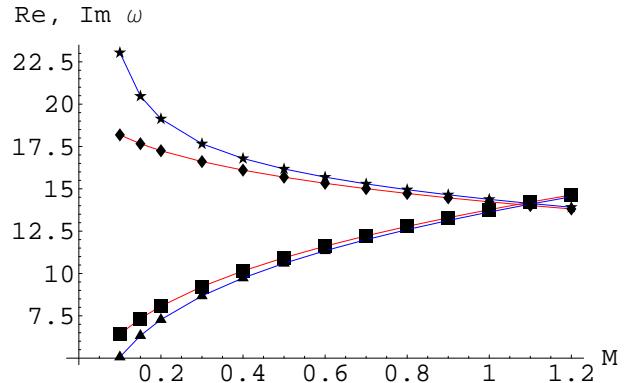


Figure 1:  $\omega_{Re}$  (top) and  $\omega_{Im}$  (bottom) parts of  $\omega$  for BTZ BH without back reaction (box, diamond) and with back reaction (star, triangle) ( $L = 2$ ,  $\Lambda = 30$ ,  $\delta = 0$ ,  $n = 0$ ) as a function of  $M$

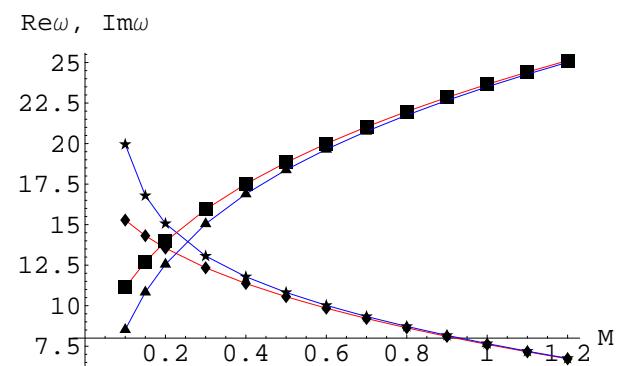


Figure 2:  $\omega_{Re}$  (top) and  $\omega_{Im}$  (bottom) parts of  $\omega$  for BTZ BH without back reaction (box, diamond) and with back reaction (star, triangle) ( $L = 2$ ,  $\Lambda = 30$ ,  $\delta = 0$ ,  $n = 1$ ) as a function of  $M$

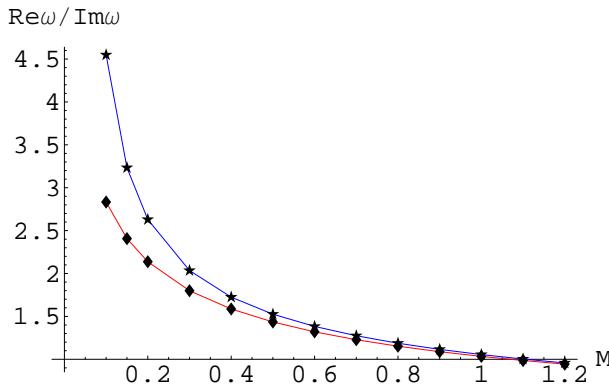


Figure 3: The ratio  $\omega_{Re}/\omega_{Im}$  for BTZ BH without back reaction (star) and with back reaction (diamond) ( $L = 2$ ,  $\Lambda = 30$ ,  $\delta = 0$ ,  $n = 0$ ) as a function of  $M$

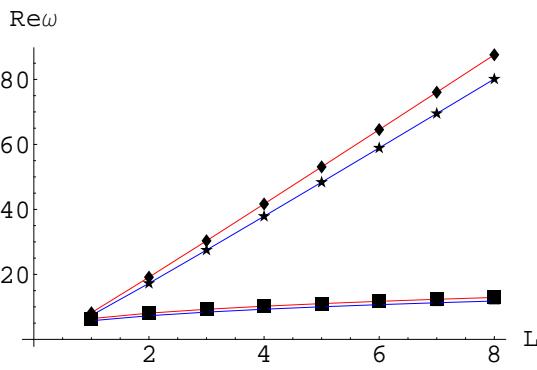


Figure 4:  $\omega_{Re}$  (top) and  $\omega_{Im}$  (bottom) parts of QNm without (box, star) and with (diamond, triangle) back reaction for different values of  $L$ , ( $M = 0.2$ ,  $\delta = 0$ ,  $n = 0$ )

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